8 Modular Forms of Higher Level

8.1 Modular Forms on $\Gamma_1(N)$

Fix integers $k \geq 0$ and $N \geq 1$. Recall that $\Gamma_1(N)$ is the subgroup of elements of $\text{SL}_2(\mathbb{Z})$ that are of the form $\left( \begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix} \right)$ when reduced modulo $N$.

**Definition 8.1.1** (Modular Forms). The space of modular forms of level $N$ and weight $k$ is

$$M_k(\Gamma_1(N)) = \{ f : f(\gamma \tau) = (c\tau + d)^k f(\tau) \text{ all } \gamma \in \Gamma_1(N) \},$$

where the $f$ are assumed holomorphic on $\mathfrak{h} \cup \{ \text{cusps} \}$ (see below for the precise meaning of this). The space of cusp forms of level $N$ and weight $k$ is the subspace $S_k(\Gamma_1(N))$ of $M_k(\Gamma_1(N))$ of modular forms that vanish at all cusps.

Suppose $f \in M_k(\Gamma_1(N))$. The group $\Gamma_1(N)$ contains the matrix $\left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$, so

$$f(z + 1) = f(z),$$

and for $f$ to be holomorphic at infinity means that $f$ has a Fourier expansion

$$f = \sum_{n=0}^{\infty} a_n q^n.$$

To explain what it means for $f$ to be holomorphic at all cusps, we introduce some additional notation. For $\alpha \in \text{GL}_2^+(\mathbb{R})$ and $f : \mathfrak{h} \to \mathbb{C}$ define another function $f_{[\alpha]}$ as follows:

$$f_{[\alpha]}(z) = \det(\alpha)^{k-1} (cz + d)^{-k} f(\alpha z).$$

It is straightforward to check that $f_{[\alpha \alpha']}_k = (f_{[\alpha]})_k [\alpha']$. Note that we do not have to make sense of $f_{[\alpha]}(\infty)$, since we only assume that $f$ is a function on $\mathfrak{h}$ and not $\mathfrak{h}^*$. 
Using our new notation, the transformation condition required for \( f : h \to C \) to be a modular form for \( \Gamma_1(N) \) of weight \( k \) is simply that \( f \) be fixed by the \([ \ ]_k\)-action of \( \Gamma_1(N) \). Suppose \( x \in \mathbb{P}^1(\mathbb{Q}) \) is a cusp, and choose \( \alpha \in \text{SL}_2(\mathbb{Z}) \) such that \( \alpha(\infty) = x \). Then \( g = f_{[\alpha]} \) is fixed by the \([ \ ]_k\) action of \( \alpha^{-1} \Gamma_1(N) \alpha \).

**Lemma 8.1.2.** Let \( \alpha \in \text{SL}_2(\mathbb{Z}) \). Then there exists a positive integer \( h \) such that \((\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}) \) is fixed by the \([ \ ]_k\) action of \( \alpha^{-1} \Gamma_1(N) \alpha \).

**Proof.** This follows from the general fact that the set of congruence subgroups of \( \text{SL}_2(\mathbb{Z}) \) is closed under conjugation by elements \( \alpha \in \text{SL}_2(\mathbb{Z}) \), and every congruence subgroup contains an element of the form \((\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix})\). If \( G \) is a congruence subgroup, then \( \Gamma(N) \) for some \( N \), and \( \alpha^{-1} \Gamma(N) \alpha = \Gamma(N) \), since \( \Gamma(N) \) is normal, so \( \Gamma(N) \subset \alpha^{-1} G \alpha \).

Letting \( h \) be as in the lemma, we have \( g(z + h) = g(z) \). Then the condition that \( f \) be holomorphic at the cusp \( x \) is that \( g(z) = \sum_{n \geq 0} b_n/h q^{1/h} \) on the upper half plane. We say that \( f \) vanishes at \( x \) if \( b_n/h = 0 \), so a cusp form is a form that vanishes at every cusp.

### 8.2 The Diamond bracket and Hecke operators

In this section we consider the spaces of modular forms \( S_k(\Gamma_1(N), \varepsilon) \), for Dirichlet characters \( \varepsilon \mod N \), and explicitly describe the action of the Hecke operators on these spaces.

#### 8.2.1 Diamond bracket operators

The group \( \Gamma_1(N) \) is a normal subgroup of \( \Gamma_0(N) \), and the quotient \( \Gamma_0(N)/\Gamma_1(N) \) is isomorphic to \( (\mathbb{Z}/N\mathbb{Z})^* \). From this structure we obtain an action of \( (\mathbb{Z}/N\mathbb{Z})^* \) on \( S_k(\Gamma_1(N)) \), and use it to decompose \( S_k(\Gamma_1(N)) \) as a direct sum of more manageable chunks \( S_k(\Gamma_1(N), \varepsilon) \).

**Definition 8.2.1** (Dirichlet character). A *Dirichlet character* \( \varepsilon \) modulo \( N \) is a homomorphism

\[
\varepsilon : (\mathbb{Z}/N\mathbb{Z})^* \to C^*.
\]

We extend \( \varepsilon \) to a map \( \varepsilon : \mathbb{Z} \to C \) by setting \( \varepsilon(m) = 0 \) if \( (m, N) \neq 1 \) and \( \varepsilon(m) = \varepsilon(m \mod N) \) otherwise. If \( \varepsilon : C \) is a Dirichlet character, the *conductor* of \( \varepsilon \) is the smallest positive integer \( \varepsilon \) that \( \varepsilon \) arises from a homomorphism \( (\mathbb{Z}/N\mathbb{Z})^* \to C^* \).

**Remarks 8.2.2.**

1. If \( \varepsilon \) is a Dirichlet character modulo \( N \) and \( M \) is a multiple of \( N \) then \( \varepsilon \) induces a Dirichlet character mod \( M \). If \( M \) is a divisor of \( N \) then \( \varepsilon \) is induced by a Dirichlet character modulo \( M \) if and only if \( M \) divides the conductor of \( \varepsilon \).
2. The set of Dirichlet characters forms a group, which is non-canonically isomorphic to $(\mathbb{Z}/N\mathbb{Z})^*$ (it is the dual of this group).

3. The mod $N$ Dirichlet characters all take values in $\mathbb{Q}(e^{2\pi i/e})$ where $e$ is the exponent of $(\mathbb{Z}/N\mathbb{Z})^*$. When $N$ is an odd prime power, the group $(\mathbb{Z}/N\mathbb{Z})^*$ is cyclic, so $e = \varphi(N)$. This double-$\varphi$ can sometimes cause confusion.

4. There are many ways to represent Dirichlet characters with a computer. I think the best way is also the simplest—fix generators for $(\mathbb{Z}/N\mathbb{Z})^*$ in any way you like and represent $\varepsilon$ by the images of each of these generators. Assume for the moment that $N$ is odd. To make the representation more “canonical”, reduce to the prime power case by writing $(\mathbb{Z}/N\mathbb{Z})^*$ as a product of cyclic groups corresponding to prime divisors of $N$. A “canonical” generator for $(\mathbb{Z}/p^r\mathbb{Z})^*$ is then the smallest positive integer $s$ such that $s \mod p^r$ generates $(\mathbb{Z}/p^r\mathbb{Z})^*$. Store the character that sends $s$ to $e^{2\pi i m/\varphi(p^r)}$ by storing the integer $n$. For general $N$, store the list of integers $n_p$, one $p$ for each prime divisor of $N$ (unless $p = 2$, in which case you store two integers $n_2$ and $n_2'$, where $n_2 \in \{0, 1\}$).

**Definition 8.2.3.** Let $\varepsilon \in (\mathbb{Z}/N\mathbb{Z})^*$ and $f \in S_k(\Gamma_1(N))$. The map $\text{SL}_2(\mathbb{Z}) \to \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective, so there exists a matrix $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N)$ such that $d \equiv \overline{d} \pmod{N}$. The *diamond bracket* $d$ operator is then

$$f(\tau)(\langle d \rangle) = f(\gamma \tau)(c\tau + d)^{-k}.$$ 

The definition of $\langle d \rangle$ does not depend on the choice of lift matrix $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$, since any two lifts differ by an element of $\Gamma(N)$ and $f$ is fixed by $\Gamma(N)$ since it is fixed by $\Gamma_1(N)$.

For each Dirichlet character $\varepsilon \mod N$ let

$$S_k(\Gamma_1(N), \varepsilon) = \{ f : f|\langle d \rangle = \varepsilon(d)f \text{ all } d \in (\mathbb{Z}/N\mathbb{Z})^* \} = \{ f : f|_{[\gamma]} = \varepsilon(d_\gamma)f \text{ all } \gamma \in \Gamma_0(N) \},$$

where $d_\gamma$ is the lower-left entry of $\gamma$.

When $f \in S_k(\Gamma_1(N), \varepsilon)$, we say that $f$ has Dirichlet character $\varepsilon$. In the literature, sometimes $f$ is said to be of “nebentypus” $\varepsilon$.

**Lemma 8.2.4.** The operator $\langle d \rangle$ on the finite-dimensional vector space $S_k(\Gamma_1(N))$ is diagonalizable.

**Proof.** There exists $N$ such that $I = \langle 1 \rangle = \langle d^n \rangle = \langle d \rangle^n$, so the characteristic polynomial of $\langle d \rangle$ divides the square-free polynomial $X^n - 1$. $\square$

Note that $S_k(\Gamma_1(N), \varepsilon)$ is the $\varepsilon(d)$ eigenspace of $\langle d \rangle$. Thus we have a direct sum decomposition

$$S_k(\Gamma_1(N)) = \bigoplus_{\varepsilon \in (\mathbb{Z}/N\mathbb{Z})^*} S_k(\Gamma_1(N), \varepsilon).$$

We have $\left( \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right) \in \Gamma_0(N)$, so if $f \in S_k(\Gamma_1(N), \varepsilon)$, then $f(\tau)(-1)^{-k} = \varepsilon(-1)f(\tau)$.

Thus $S_k(\Gamma_1(N), \varepsilon) = 0$, unless $\varepsilon(-1) = (-1)^k$, so about half of the direct summands $S_k(\Gamma_1(N), \varepsilon)$ vanish.
8.2.2 Hecke Operators on $q$-expansions

Suppose

$$f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_1(N), \varepsilon),$$

and let $p$ be a prime. Then

$$f|_{T_p} = \begin{cases} 
\sum_{n=1}^{\infty} a_{np} q^n + p^{k-1} \varepsilon(p) \sum_{n=1}^{\infty} a_n q^{pn}, & p \nmid N \\
\sum_{n=1}^{\infty} a_{np} q^n + 0, & p \mid N.
\end{cases}$$

Note that $\varepsilon(p) = 0$ when $p \mid N$, so the second part of the formula is redundant.

When $p \mid N$, $T_p$ is often denoted $U_p$ in the literature, but we will not do so here. Also, the ring $T$ generated by the Hecke operators is commutative, so it is harmless, though potentially confusing, to write $T_p(f)$ instead of $f|_{T_p}$.

We record the relations

$$T_m T_n = T_{mn}, \quad (m, n) = 1,$$

$$T_{p^k} = \begin{cases} 
(T_p)^k, & p \mid N \\
T_{p^{k-1}} T_p - \varepsilon(p) p^{k-1} T_{p^{k-2}}, & p \nmid N.
\end{cases}$$

**WARNING:** When $p \mid N$, the operator $T_p$ on $S_k(\Gamma_1(N), \varepsilon)$ need not be diagonalizable.

8.3 Old and new subspaces

Let $M$ and $N$ be positive integers such that $M \mid N$ and let $t \mid \frac{N}{M}$. If $f(\tau) \in S_k(\Gamma_1(M))$ then $f(t\tau) \in S_k(\Gamma_1(N))$. We thus have maps

$$S_k(\Gamma_1(M)) \to S_k(\Gamma_1(N))$$

for each divisor $t \mid \frac{N}{M}$. Combining these gives a map

$$\varphi_M : \bigoplus_{t \mid \frac{N}{M}} S_k(\Gamma_1(M)) \to S_k(\Gamma_1(N)).$$

**Definition 8.3.1** (Old Subspace). The *old subspace* of $S_k(\Gamma_1(N))$ is the subspace generated by the images of the $\varphi_M$ for all $M \mid N$ with $M \neq N$.

**Definition 8.3.2** (New Subspace). The *new subspace* of $S_k(\Gamma_1(N))$ is the complement of the old subspace with respect to the Petersson inner product.

\footnote{Since I haven’t introduced the Petersson inner product yet, note that the new subspace of $S_k(\Gamma_1(N))$ is the largest subspace of $S_k(\Gamma_1(N))$ that is stable under the Hecke operators and has trivial intersection with the old subspace of $S_k(\Gamma_1(N))$.}

\footnote{Remove from book.}
8.3 Old and new subspaces

**Definition 8.3.3 (Newform).** A *newform* is an element $f$ of the new subspace of $S_k(\Gamma_1(N))$ that is an eigenvector for every Hecke operator, which is normalized so that the coefficient of $q$ in $f$ is 1.

If $f = \sum a_n q^n$ is a newform then the coefficients $a_n$ are algebraic integers, which have deep arithmetic significance. For example, when $f$ has weight 2, there is an associated abelian variety $A_f$ over $\mathbb{Q}$ of dimension $[\mathbb{Q}(a_1, a_2, \ldots) : \mathbb{Q}]$ such that $\prod L(f^\sigma, s) = L(A_f, s)$, where the product is over the $\text{Gal}(\mathbb{Q}/\mathbb{Q})$-conjugates of $f$. The abelian variety $A_f$ was constructed by Shimura as follows. Let $J_1(N)$ be the Jacobian of the modular curve $X_1(N)$. As we will see tomorrow, the ring $T$ of Hecke operators acts naturally on $J_1(N)$. Let $I_f$ be the kernel of the homomorphism $T \to \mathbb{Z}[a_1, a_2, \ldots]$ that sends $T_n$ to $a_n$. Then

$$A_f = J_1(N)/I_f J_1(N).$$

In the converse direction, it is a deep theorem of Breuil, Conrad, Diamond, Taylor, and Wiles that if $E$ is any elliptic curve over $\mathbb{Q}$, then $E$ is isogenous to $A_f$ for some $f$ of level equal to the conductor $N$ of $E$.

When $f$ has weight greater than 2, Scholl constructs\(^2\), in an analogous way, a Grothendieck motive \(^=\text{compatible collection of cohomology groups}\(^3\)) $M_f$ attached to $f$.

\(^2\)add reference
\(^3\)remove